

# JACOBIANS WITH A VANISHING THETA-NULL IN GENUS 4

SAMUEL GRUSHEVSKY AND RICCARDO SALVATI MANNI

**ABSTRACT.** In this paper we prove a conjecture of Hershel Farkas [8] that if a 4-dimensional principally polarized abelian variety has a vanishing theta-null, and the hessian of the theta function at the corresponding point of order two is degenerate, the abelian variety is a Jacobian.

We also discuss possible generalizations to higher genera, and an interpretation of this condition as an infinitesimal version of Andreotti and Mayer's local characterization of Jacobians by the dimension of the singular locus of the theta divisor.

## INTRODUCTION

The study of the geometry of the theta divisor of Jacobians of algebraic curves is a very classical subject going back at least to Riemann's celebrated theta-singularity theorem itself, later strengthened by Kempf [17]. The geometry of the canonical curve was also shown to be related to the geometry of the theta divisor — in particular the celebrated Green's conjecture [13] says that the tangent cones to the Jacobian theta divisor at its double points span the ideal of quadrics containing the canonical curve.

These tangent cone quadrics in fact have rank 4; it can also be shown that the singular locus of the theta divisor  $\text{Sing } \Theta$  of a Jacobian of a curve of genus  $g$  has dimension  $\geq g - 4$  ( $g - 3$  for hyperelliptic curves). It was thus asked whether this is a characteristic property for the locus of Jacobians  $\mathcal{J}_g$  within the moduli space of principally polarized abelian varieties.

This study was undertaken by Andreotti and Mayer [2]. Let  $\mathcal{A}_g$  be the moduli space of (complex) principally polarized abelian varieties — ppavs for short. Denote by  $N_k \subset \mathcal{A}_g$  the locus of ppavs for which  $\dim \text{Sing } \Theta \geq k$ . Andreotti and Mayer showed that  $\mathcal{J}_g$  is an irreducible component of  $N_{g-4}$ . The situation was further studied by Beauville [3] and Debarre [5]. It was shown that for  $g \geq 4$  the locus  $N_{g-4}$  is reducible (and thus not equal to  $\mathcal{J}_g$ ); however, conjecturally all components of  $N_{g-4}$  other than  $\mathcal{J}_g$  are contained in the theta-null divisor  $\theta_{\text{null}}$  (the

zero locus of the product of all theta constants with half-integral characteristics; alternatively, the locus of those ppavs for which the theta divisor has a singularity at a point of order two), which is a component of  $N_0$ .

Thus it is natural to try to study the intersection of  $\mathcal{J}_g$  with the other components of  $N_{g-4}$ , or at least with  $\theta_{\text{null}} \subset N_0$ . This is the object of this paper: we study the infinitesimal version of  $N_{g-4}$ , prove H. Farkas' conjecture [8] describing the situation for  $g = 4$ , and discuss the possible situation in general.

In a recent paper [14], we proved some identities between the simplest types of theta series with harmonic polynomial coefficients, which are generalizations of the classical Jacobi's derivative formula. Proving these involves the evaluation at zero of the derivatives of first and second order of classical theta functions, i.e. some local infinitesimal properties of the theta divisor. The methods developed there are applicable to our problem and provide a natural generalization of Farkas' question as follows.

Consider the following stratification of  $\theta_{\text{null}}$ : let  $\theta_{\text{null}}^h \subset \theta_{\text{null}}$  be the subset where the tangent cone to the theta divisor at the corresponding singular point of order two has rank  $\leq h$ , for  $h = 0, 1, \dots, g$ . We have equations for the (level covers of) loci  $\theta_{\text{null}}^h$ , and believe that a further investigation of the relation between  $\mathcal{J}_g \cap \theta_{\text{null}}$  and  $\theta_{\text{null}}^3$  would be of interest. Similarly it seems promising to study the locus  $\theta_{\text{null}}^2$  locally near the locus of reducible ppavs (which is contained in it). In this paper we treat the  $g = 4$  case completely; higher dimensions will be addressed in a separate paper.

## 1. NOTATIONS AND DEFINITIONS

**Definition 1.** Let  $\mathcal{H}_g$  denote the *Siegel upper half-space*, i.e. the set of symmetric complex  $g \times g$  matrices  $\tau$  with positive definite imaginary part. Each such  $\tau$  defines a complex principally polarized abelian variety (ppav for short)  $A_\tau := \mathbb{C}^g / \mathbb{Z}^g + \tau \mathbb{Z}^g$ . If  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z})$  is a symplectic matrix in a  $g \times g$  block form, then its action on  $\tau \in \mathcal{H}_g$  is defined by  $M \circ \tau := (a\tau + b)(c\tau + d)^{-1}$ , and the moduli space of ppavs is the quotient  $\mathcal{A}_g = \mathcal{H}_g / \mathrm{Sp}(2g, \mathbb{Z})$ .

A period matrix  $\tau$  is called *reducible* if there exists  $M \in \mathrm{Sp}(2g, \mathbb{Z})$  such that

$$M \cdot \tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \tau_i \in \mathcal{H}_{g_i}, \quad g_1 + g_2 = g;$$

otherwise we say that  $\tau$  is irreducible.

**Definition 2.** For  $\varepsilon, \delta \in (\mathbb{Z}/2\mathbb{Z})^g$ , thought of as vectors of zeros and ones,  $\tau \in \mathcal{H}_g$  and  $z \in \mathbb{C}^g$ , the *theta function with characteristic*  $[\varepsilon, \delta]$  is

$$\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau, z) := \sum_{m \in \mathbb{Z}^g} \exp \pi i \left[ {}^t(m + \frac{\varepsilon}{2})\tau(m + \frac{\varepsilon}{2}) + 2 {}^t(m + \frac{\varepsilon}{2})(z + \frac{\delta}{2}) \right].$$

A *characteristic*  $[\varepsilon, \delta]$  is called *even* or *odd* depending on whether  $\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau, z)$  is even or odd as a function of  $z$ , which corresponds to the scalar product  $\varepsilon \cdot \delta \in \mathbb{Z}/2\mathbb{Z}$  being zero or one, respectively. A *theta constant* is the evaluation at  $z = 0$  of a theta function. All odd theta constants of course vanish identically in  $\tau$ .

Observe that

$$\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left( \tau, z + \tau \frac{\varepsilon}{2} + \frac{\delta}{2} \right) = \exp \pi i \left( -\frac{{}^t\varepsilon}{2} \tau \frac{\varepsilon}{2} - \frac{{}^t\varepsilon}{2} (z + \frac{\delta}{2}) \right) \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau, z),$$

i.e. theta functions with characteristics are, up to some easy factor, the Riemann's theta function (the one with characteristic  $[0, 0]$ ) shifted by points of order two.

Let  $\rho : \mathrm{GL}(g, \mathbb{C}) \rightarrow \mathrm{End} V$  be an irreducible rational representation with the highest weight  $(k_1, k_2, \dots, k_g)$ ,  $k_1 \geq k_2 \geq \dots \geq k_g$ ; then we call  $k_g$  the weight of  $\rho$ . A representation  $\rho_0$  is called reduced if its weight is equal to zero. Let us fix an integer  $r$ ; we are interested in pairs  $\rho = (\rho_0, r)$ , with  $\rho_0$  reduced. We call  $r$  the weight of  $\rho$  and use the notation

$$\rho(A) = \rho_0(A) \det A^{r/2}.$$

**Definition 3.** A map  $f : \mathcal{H}_g \rightarrow V$  is called a  *$\rho$ -valued modular form* with respect to a finite index subgroup  $\Gamma \subset \mathrm{Sp}(2g, \mathbb{Z})$  if

$$f(\sigma \circ \tau) = v(\sigma) \rho(c\tau + d) f(\tau) \quad \forall \tau \in \mathcal{H}_g, \forall \sigma \in \Gamma,$$

and if additionally  $f$  is holomorphic at all cusps of  $\mathcal{H}_g/\Gamma$ .

If  $\rho(\sigma) = \det(c\tau + d)^k$ , then we call this a *scalar modular form* of weight  $k$ .

For a finite index subgroup  $\Gamma \subset \mathrm{Sp}(2g, \mathbb{Z})$  a multiplier system of weight  $r/2$  is a map  $v : \Gamma \rightarrow \mathbb{C}^*$ , such that the map

$$\sigma \mapsto v(\sigma) \det(c\tau + d)^{r/2}$$

satisfies the cocycle condition for every  $\sigma \in \Gamma$  and  $\tau \in \mathcal{H}_g$  (note that the function  $\det(c\tau + d)$  possesses a square root). Clearly a multiplier system of integral weight is a character.

**Definition 4.** A map  $f : \mathcal{H}_g \rightarrow V$  is called a  $\rho$ - or  $V$ -valued modular form, or simply a vector-valued modular form, if the choice of  $\rho$  is clear, with multiplier  $v$ , with respect to a finite index subgroup  $\Gamma \subset \mathrm{Sp}(2g, \mathbb{Z})$  if

$$(1) \quad f(\sigma \circ \tau) = v(\sigma)\rho(c\tau + d)f(\tau) \quad \forall \tau \in \mathcal{H}_g, \forall \sigma \in \Gamma,$$

and if additionally  $f$  is holomorphic at all cusps of  $\mathcal{H}_g/\Gamma$ .

**Definition 5** (Theta constants are modular forms). We define the *level* subgroups of the symplectic group to be

$$\Gamma_g(n) := \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z}) \mid M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n} \right\}$$

$$\Gamma_g(n, 2n) := \left\{ M \in \Gamma_g(n) \mid \mathrm{diag}(a^t b) \equiv \mathrm{diag}(c^t d) \equiv 0 \pmod{2n} \right\}.$$

Under the action of  $M \in \mathrm{Sp}(2g, \mathbb{Z})$  the theta functions transform as follows:

$$\begin{aligned} \theta \left[ M \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} \right] (M \cdot \tau, {}^t(c\tau + d)^{-1}z) \\ = \phi(\varepsilon, \delta, M, \tau, z) \det(c\tau + d)^{\frac{1}{2}} \theta \left[ \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} \right] (\tau, z), \end{aligned}$$

where

$$M \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} := \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} + \begin{pmatrix} \mathrm{diag}(c^t d) \\ \mathrm{diag}(a^t b) \end{pmatrix}$$

taken modulo 2, and  $\phi(\varepsilon, \delta, M, \tau, z)$  is some complicated explicit function. For more details, we refer to [15] and [9].

Thus theta constants with characteristics are (scalar) modular forms of weight 1/2 with respect to  $\Gamma_g(4, 8)$ , i.e. we have

$$\theta \left[ \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} \right] (M \circ \tau, 0) = \det(c\tau + d)^{1/2} \theta \left[ \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} \right] (\tau, 0) \quad \forall M \in \Gamma_g(4, 8).$$

**Definition 6.** We call the *theta-null divisor*  $\theta_{\text{null}} \subset \mathcal{A}_g$  the zero locus of the product of all even theta constants. We define a stratification of  $\theta_{\text{null}}$  as follows. For  $h = 0, \dots, g$  we let

$$\theta_{\text{null}}^h = \left\{ \tau \in \mathcal{H}_g : \exists [\varepsilon, \delta] \text{ even}, \theta \left[ \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} \right] (\tau) = 0; \left. \mathrm{rk} \frac{\partial^2 \theta \left[ \begin{pmatrix} \varepsilon \\ \delta \end{pmatrix} \right] (\tau, z)}{\partial z_i \partial z_j} \right|_{z=0} \leq h \right\},$$

i.e. the locus of points on  $\theta_{\text{null}}$  where the rank of the tangent cone to the theta divisor at the corresponding point  $\frac{\tau\varepsilon+\delta}{2}$  of order two is at most  $h$ .

By the above transformation formulae, we see that  $\theta_{\text{null}}$  and  $\theta_{\text{null}}^h$  are well-defined on  $\mathcal{A}_g$  and not only on the *level moduli spaces*  $\mathcal{A}_g(4, 8) := \mathcal{H}_g/\Gamma_g(4, 8)$ . Since the theta constant with characteristic is up to a non-zero factor the value of Riemann's theta function at the corresponding point of order two,  $\theta_{\text{null}}$  can also be described as the locus of ppavs for which  $\Theta$  has a singularity at a point of order two. Note that  $\theta(\tau, z)$  is an even function of  $z$  (the theta divisor is symmetric under the involution  $\pm 1$ ), and thus the locus  $\theta_{\text{null}}$  indeed turns out to be a divisor).

**7** (The ring of modular forms). We recall that the theta constants define an embedding

$$\begin{aligned} Th : \mathcal{A}_g(4, 8) &\rightarrow \mathbb{P}^{2^{g-1}(2^g+1)-1} \\ \tau &\mapsto \left\{ \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau) \right\}_{[\varepsilon, \delta] \text{ even}}. \end{aligned}$$

This map extends to the Satake compactification  $\overline{\mathcal{A}_g(4, 8)}$ . Hence the ring of scalar modular forms for  $\Gamma(4, 8)$  is the integral closure of the ring  $\mathbb{C} \left[ \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} \right]$ . The ideal of algebraic equations defining  $Th(\overline{\mathcal{A}_g(4, 8)}) \subset \mathbb{P}^{2^{g-1}(2^g+1)-1}$  is known completely only for  $g \leq 2$  (and almost known for  $g = 3$ , see [11]).

**8.** Since theta functions satisfy the heat equation

$$\frac{\partial^2 \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau, z)}{\partial z_i \partial z_j} = \pi i (1 + \delta_{ij}) \frac{\partial \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau, z)}{\partial \tau_{ij}},$$

(where  $\delta_{ij}$  is Kronecker's delta), the Hessian of the theta functions with respect to  $z$  can be rewritten as the first derivatives with respect to  $\tau_{ij}$ . Hence if a point  $x = \tau \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} + \frac{1}{2}$  of order two is a singular point in the theta divisor, which is simply to say  $\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau, x) = 0 = \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau, 0)$  (the first derivatives at zero of an even function are all zero), the rank of the quadric defining the tangent cone at  $x$  is the rank of the matrix obtained by applying the  $g \times g$ -matrix-valued differential operator

$$\mathcal{D} := \begin{pmatrix} \frac{\partial}{\partial \tau_{11}} & \frac{1}{2} \frac{\partial}{\partial \tau_{12}} & \cdots & \frac{1}{2} \frac{\partial}{\partial \tau_{1g}} \\ \frac{1}{2} \frac{\partial}{\partial \tau_{21}} & \frac{\partial}{\partial \tau_{22}} & \cdots & \frac{1}{2} \frac{\partial}{\partial \tau_{2g}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2} \frac{\partial}{\partial \tau_{g1}} & \cdots & \cdots & \frac{\partial}{\partial \tau_{gg}} \end{pmatrix}$$

to  $\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau, 0)$ .

## 2. EQUATIONS FOR $\theta_{\text{null}}^h$

The locus  $\theta_{\text{null}}^h$  is given by the conditions

$$\{\exists [\varepsilon, \delta] \text{ even; } 0 = \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau); \text{ rk } \mathcal{D}\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau) \leq h\}.$$

We can get equations for  $\theta_{\text{null}}^h$  by setting all  $(h+1) \times (h+1)$  minors of  $\mathcal{D}\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau)$  equal to zero, but these minors are not modular forms: the derivative of a section of a bundle is only a section of that bundle when restricted to the zero set of the section, i.e.  $\mathcal{D}\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau)$  is not a modular form, but is modular when restricted to the locus  $\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau) = 0$ .

This condition is not invariant under  $\text{Sp}(2g, \mathbb{Z})/\Gamma_g(4, 8)$ , and thus for technical reasons we will work on  $\mathcal{A}_g(4, 8)$ . However, the locus  $\theta_{\text{null}}^h$  that we are describing is invariant under the action of  $\text{Sp}(2g, \mathbb{Z})$ , and we will thus be able to easily descend from  $\mathcal{A}_g(4, 8)$  to  $\mathcal{A}_g$  by symmetrizing.

The divisor  $\theta_{\text{null}} \subset \mathcal{A}_g(4, 8)$  is reducible. Its irreducible components are the divisors of individual theta constants with characteristics — cf. [10] page 88 for  $g \geq 3$  and by inspection in the remaining two cases. These components are all conjugate under the action of  $\text{Sp}(2g, \mathbb{Z})$ , and thus for our purposes we can restrict to one component. Without loss of generality we can take this to be  $\theta_0 := \{\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) = 0\} \subset \mathcal{A}_g(4, 8)$ ,

and consider its stratification, letting  $\theta_0^h$  be those  $\tau \in \theta_0$  for which the rank of the tangent cone, i.e. of the hessian of the theta function, at zero is at most  $h$ .

Following the ideas of [14], we observe that for any even characteristic  $[\varepsilon, \delta] \neq [0, 0]$  the expression

$$\begin{aligned} & \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau)^2 \mathcal{D}(\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}/\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix})(\tau) \\ &= (1 + \delta_{i,j}) \left[ \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau) \frac{\partial \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau)}{\partial \tau_{ij}} - \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) \frac{\partial \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau)}{\partial \tau_{ij}} \right] \end{aligned}$$

is a vector-valued modular form of weight 1 with  $\rho_0 = (2, 0, \dots, 0)$  with respect to  $\Gamma_g(4, 8)$ .

We denote by  $\mathcal{B}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}\right)^h(\tau)$  the  $\binom{g}{h} \times \binom{g}{h}$  symmetric matrix obtained by taking in lexicographic order all the  $h \times h$  of the above matrix — it is a vector valued modular form of weight  $h$  with  $\rho_0 = (2, \dots, 2, 0, \dots, 0)$ , with respect to  $\Gamma_g(4, 8)$ .

**Theorem 9.** *The locus  $\theta_0^h$  is set theoretically defined by*

$$\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau, 0) = 0 = \mathcal{B}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}\right)^{h+1}(\tau) \quad \forall [\varepsilon, \delta] \neq [0, 0] \text{ even.}$$

*Proof.* One implication is trivial. Viceversa, let us assume that all equations are satisfied; there always exists an even characteristic  $[\varepsilon, \delta]$  such that  $\theta\begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau) \neq 0$ , thus  $\mathcal{B}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}\right)^{h+1}(\tau) = 0$  implies  $\tau \in \theta_0^h$ .  $\square$

### 3. INFINITESIMAL ANDREOTTI-MAYER CONDITION

We would like to understand the loci  $\theta_{\text{null}}^h$  and  $\theta_{\text{null}} \cap \mathcal{J}_g$ . Suppose we have a Jacobian with a vanishing theta-null, i.e.  $X \in \mathcal{J}_g \cap \theta_{\text{null}}$ , and  $x \in X[2]$  is the corresponding point of order two on the theta divisor. It is a consequence of Kempf's singularity theorem (see [17], [1]) that the rank of the hessian of the theta function (i.e. of the tangent cone) at  $x$  is equal to three. By Green's conjecture [13] the quadrics defining the canonical curve are obtained exactly as Hessians of the theta function with respect to  $z$  at singular points of the theta divisor. In general they have rank at most 4, but it is easy to see that at singular points of order two the rank actually drops to three — see [1] and also [8]. Thus  $X$  belongs to  $\theta_{\text{null}}^3$ , so we see that  $(\mathcal{J}_g \cap \theta_{\text{null}}) \subset \theta_{\text{null}}^3$ .

It is natural to view the Hessian rank condition  $\theta_{\text{null}}^h$  as the infinitesimal version of the Andreotti-Mayer condition, indicating that  $\text{Sing } \Theta$  has a  $(g - 4)$ -dimensional 2-jet at  $x$ .

The codimension of the space of  $g \times g$  symmetric matrices of rank at most 3 within  $\mathcal{H}_g$  is equal to  $(g - 1)(g - 2)/2$  (the dimension is  $3g - 3$ : choosing the first three rows, and thus also the first three columns, generically determines everything), and thus for dimension reasons it is natural to expect that locally the condition of the rank being 3 characterizes Jacobians with a vanishing theta-null. We believe that the infinitesimal Andreotti-Mayer condition at points of order two should help to characterize Jacobians with a vanishing theta-null locally. We thank Enrico Arbarello, Hershel Farkas and Edoardo Sernesi for stimulating discussions on this topic.

To study this link, one could try to “integrate” the local condition, to show that  $\text{Sing } \Theta$ , indeed, has dimension  $g - 4$ , but this seems hard. By using the ideas of [2] (see also [1]), we can get for  $\tau \in \theta_{\text{null}}^3$  some equations involving the second derivative of the theta functions with respect to the variable  $\tau$ , but we could not give a good dimensional estimate of the conditions thus obtained.

Another potential approach to this problem would be to use theorem 6 from [14] to express the  $\mathcal{B}$  in terms of Jacobian determinants of odd theta functions, with syzygetic characteristics. If one could then prove the appropriate generalization of Jacobi’s derivative formula, conjectured for all genera and proven for genus 4 in [19], this expression in terms of Jacobian determinants could then be rewritten as an algebraic expression in terms of theta constants, and then could perhaps be compared to Schottky-Jung [9] equations for theta constants, or could be used to give a conjectural local algebraic solution to the Schottky problem.

One could also try to study the infinitesimal Andreotti-Mayer condition at singular points of the theta divisor that are *not* points of order two, and see whether the rank of the Hessian there can be used to locally characterize the Jacobian locus. In genus 4 there is no problem, since  $N_0 = \theta_{\text{null}} \cup \mathcal{J}_4$ , but the situation in higher dimensions seems very complicated, as it is not clear how to interpret such a condition in terms of modular forms.

It is known that  $N_{g-2}$  is the locus of reducible ppavs [7]. It is then clear that ppavs with reducible theta divisor are in  $\theta_{\text{null}}^2$  — in this case the tangent cone is a quadric that is the union of two hyperplanes. We are naturally led to consider the link between  $N_{g-2}$  and  $\theta_{\text{null}}^2$ . This will be done in a forthcoming paper.

#### 4. THE THEOREM IN GENUS 4

Our main result of this section is the proof of the following conjecture of H. Farkas [8]:

**Theorem 10.** *If for a 4-dimensional ppav  $\tau \in \mathcal{A}_4$  some theta constant and its hessian are both equal to zero, this ppav is a Jacobian, i.e.*

$$\theta_{\text{null}}^3 = \mathcal{J}_4 \cap \theta_{\text{null}}.$$

*Proof.* The fact that this vanishing holds for Jacobians (i.e. the implication  $\Leftarrow$ ) is discussed in the previous section.

Since  $\text{Sp}(4, \mathbb{Z})/\Gamma_4(4, 8)$  acts transitively on the set of theta constants with characteristics, it is enough to take  $[\varepsilon, \delta] = [0, 0]$  above, and show

(here  $\theta$  denotes the theta function with zero characteristics) that if  $\theta(\tau) = \det \mathcal{D}\theta(\tau) = 0$ , then  $\tau \in \mathcal{J}_4 \cap \theta_{\text{null}}$ .

We denote by  $\mathcal{J}_4(4, 8) \subset \mathcal{A}_4(4, 8)$  the Jacobian locus; also denote  $J := Th(\mathcal{J}_4(4, 8)) \subset A := Th(\mathcal{A}_4(4, 8))$ , and denote the chosen component of the theta-null by  $T := A \cap \{\theta(\tau) = 0\}$ . Let us denote the locus we are interested in by  $S := T \cap \{\det \mathcal{D}\theta(\tau) = 0\}$ .

The main theorem is then the statement that  $S = J \cap T$ , of which we already know the inclusion  $S \supset J \cap T$  from the previous discussion.

**Lemma 11.**  $\dim S = \dim(J \cap T) = 8$ .

*Proof.* It is known that no theta constant vanishes identically on the Jacobian locus. Thus  $J \cap T \not\subseteq J$ , and is given by one equation, so  $\dim(J \cap T) = \dim J - 1 = 8$ . On the other hand, the locus  $\{\det \mathcal{D}\theta(\tau) = 0\} \subset \mathcal{H}_4$  does not contain  $\{\theta(\tau) = 0\} \subset \mathcal{H}_4$ , since they are both of codimension 1 in  $\mathcal{H}_4$ , and the first locus is not invariant under  $\Gamma_4(4, 8)$ , while the second is. Thus we have  $S \not\subseteq T$ , and since locally in  $\mathcal{H}_g$  it is given by one extra equation, we get  $\dim S = \dim T - 1 = \dim A - 2 = 8$ .  $\square$

By the discussion in the previous section we know that  $S \supset J \cap T$ , and to prove that  $S = J \cap T$  it is enough to show that the degrees of the two sets, as of 8-dimensional subvarieties of the projective space, are equal. We will now compute these degrees.

**Lemma 12.** *The set-theoretic degree  $\deg(J \cap T) = 8 \deg A$ . Scheme-theoretically, the degree of this intersection is  $16 \deg A$ .*

*Proof.* Since  $\{\theta(\tau) = 0\}$  is a hyperplane in  $\mathbb{P}^{135}$ , we have  $\deg T = \deg A$ , and  $\deg(J \cap T) = \deg J$ . Recalling that  $A$  and  $J$  are both irreducible varieties, and  $J \subset A$  is given by one equation of degree 16 in theta constants, i.e. by a polynomial of degree 16 [11] in the coordinates of  $\mathbb{P}^{135}$ , we see that  $\deg J = 16 \deg A$  scheme-theoretically.

However, the equation for  $J \subset A$  in the notations of [11] is

$$(r_1 + r_2 + r_3)(r_1 - r_2 + r_3)(r_1 + r_2 - r_3)(r_1 - r_2 - r_3) = 0,$$

where each  $r_i$  is a monomial of degree 4 in theta constants.<sup>1</sup> If a theta constant vanishes, i.e. if we are on the theta-null divisor, it means that one of the products  $r_i$  vanishes, so without loss of generality let's say

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<sup>1</sup>There are many possible choices for  $r_i$ , and thus many resulting forms of the equation, which are all conjugate under the action of  $Sp(4, \mathbb{Z})/\Gamma_4(4, 8)$ . What happens is that  $A \subset \mathbb{P}^{135}$  is itself given by a large number of equations (explicitly unknown to this date), and when we intersect  $A$  with *any* equation of the above form, the intersection is always the same, and in particular invariant under  $Sp(4, \mathbb{Z})/\Gamma_4(4, 8)$ .

$r_3 = 0$ . In this case the above equation becomes  $(r_1 - r_2)^2(r_1 + r_2)^2 = 0$ , so that there is multiplicity at least two. To finish the proof of the lemma, we need to show that the multiplicity is exactly two, for which we will use the argument similar to the one used in [16] to prove local irreducibility of Schottky's divisor.

Indeed, take a diagonal period matrix  $\tau_0 \in \mathcal{H}_1^{\times 4}$  with diagonal entries  $\omega_1, \omega_2, \omega_3, \omega_4$ , and denote by  $\mathcal{O}$  the analytic local ring of  $\mathcal{H}_4$  at  $\tau_0$ . We shall use  $\tau_{ii} - \omega_i$  for  $1 \leq i \leq 4$ , and  $2\pi\sqrt{-1}\tau_{ij}$  for  $1 \leq i < j \leq 4$  as the generators of the maximal ideal  $\mathfrak{m} \subset \mathcal{O}$ . If we arrange the  $2\pi\sqrt{-1}\tau_{ij}$  in the order (12), (34), (13), (24), (14), (23) and call them  $x_1, x_2, \dots, x_6$ , then the expression  $(r_1 + r_2 + r_3)(r_1 - r_2 + r_3)(r_1 + r_2 - r_3)(r_1 - r_2 - r_3)$  in  $\mathfrak{m}^8/\mathfrak{m}^9$  is equal to

$$2^{16}\delta(\omega_1)\delta(\omega_2)\delta(\omega_3)\delta(\omega_4)P(X),$$

where  $\delta(\omega)$  is the unique cusp form of weight 12 for  $\mathrm{SL}(2, \mathbb{Z})$ , suitably normalized, cf. [16] and

$$P(X) = (x_1x_2 - x_3x_4)^2(x_5x_6)^2 - 2(x_1x_2 + x_3x_4) \prod_{i=1}^6 x_i + \left( \prod_{i=1}^4 x_i \right)^2.$$

For simplicity choose, for this lemma only, the vanishing theta constant to be  $\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau)$  with  $\varepsilon = \delta = (1100)$ , instead of the zero characteristic — this theta constant in  $\mathfrak{m}/\mathfrak{m}^2$  has local expression  $x_1$ . Hence, by substitution, we have the expression  $(x_3x_4x_5x_6)^2$  for  $P(X)$ , so the multiplicity of the intersection of  $T$  and  $J$  (as given by the Schottky relation) is exactly two. As a consequence, the set-theoretic degree is half of the scheme-theoretic degree.  $\square$

To compute  $\deg S$ , we use results from the previous sections. Our original problem in dealing with  $S$  is that  $\det \mathcal{D}(\theta(\tau))$  is not a modular form, and thus its zero locus is not well-defined on  $\mathcal{A}_g(4, 8)$ . However, we can apply theorem 9 for  $g = 4, h = 3$ .

**Lemma 13.** *Define a function  $F$  on  $\mathcal{H}_g$  by*

$$F(\tau) := \left( \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau) \right)^{2g} \det \mathcal{D} \left( \frac{\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau)}{\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau)} \right).$$

*Then for any even  $[\varepsilon, \delta]$ , scheme-theoretically we have*

$$T \cap \{F(\tau) = 0\} = T \cap \left\{ 0 = \left( \theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau) \right)^g \det \mathcal{D}\theta(\tau) \right\}$$

*Proof.* This is obtained by writing out the derivatives involved in  $F$  and using  $\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) = 0$  on  $T$ .  $\square$

As discussed in [14],  $F(\tau)$  is a scalar modular form of weight  $g + 2$  with respect to  $\Gamma_g(4, 8)$  (recall that theta constants have weight  $1/2$ ). We can now compute the degree of  $S$  and thus finish the proof of the theorem.

**Lemma 14.** *The set-theoretic degree  $\deg S = 8 \deg A (= \deg(J \cap T))$ .*

*Proof.* For  $g = 4$  the form  $F$  is of weight 6, and thus it follows that it is a section of the 12'th power of the bundle of theta constants, the degree of the zero locus of  $F$  in  $\mathbb{P}^{135}$  is equal to 12. Thus we have  $\deg(T \cap \{F(\tau) = 0\}) = 12 \deg T = 12 \deg A$ .

To understand the locus  $T \cap \{F(\tau) = 0\}$ , note that on the right-hand-side of lemma 13 we have the union of two loci:  $S$ , which is exactly  $T \cap \{\det \mathcal{D}\theta(\tau) = 0\}$ , and of  $T \cap \{\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix}(\tau) = 0\}$ , with multiplicity  $g = 4$ . Of course the latter is the intersection of  $A$  with two hyperplanes, so its degree is equal to  $\deg A$ . Thus comparing the scheme-theoretic degrees on both sides of the equality in the previous lemma, we get the scheme-theoretic degree  $\deg S = 12 \deg A - 4 \deg A = 8 \deg A$ . Since  $S \supset (J \cap T)$  of the same dimension, which already has degree  $8 \deg A$ , there is no multiplicity, and the set-theoretic degree of  $S$  is the same as scheme-theoretic.  $\square$

Since we have shown that  $\deg S = \deg J \cap T$  as of sets, and we know  $S \supset (J \cap T)$ , the theorem — the statement  $S = J \cap T$  — finally follows.  $\square$

As an immediate consequence we have the following

**Corollary 15.** *If for  $\tau \in \mathcal{A}_4$  some theta constants and all minors of order 3 of its hessian are equal to zero, this ppav has reducible theta divisor.*

*Proof.* By the theorem we know that  $\tau \in \overline{\mathcal{J}}_4$ . Since the tangent cone has rank two, it is reducible, and thus  $\tau$  is a reducible point.  $\square$

Note that in trying to approach the problem in higher genus one trouble with this corollary is that the locus of reducible ppavs within  $\mathcal{A}_g$  is not contained in the closure of the Jacobian locus, while for genus 4 this is the case, as we have  $\overline{\mathcal{J}}_g = \mathcal{A}_g$  for  $g = 1, 2, 3$ .

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MATHEMATICS DEPARTMENT, PRINCETON UNIVERSITY, FINE HALL, WASHINGTON ROAD, PRINCETON, NJ 08544, USA

*E-mail address:* sam@math.princeton.edu

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ “LA SAPIENZA”, PIAZZALE A. MORO 2, ROMA, I 00185, ITALY

*E-mail address:* salvati@mat.uniroma1.it